Moduli of boundary-polarized Calabi-Yau pairs:

with Ascher, Bejleri, Blum, DeVleming, Liu, Wang; $/\mathbb{C}$

||How to compactify moduli of pairs||

$\mathcal{U} = \{(X,D)\}$ moduli space of pairs, $\mathcal{U}$ not compact, then

① $K_X + D$ ample $\rightsquigarrow$ KSBA stability one can compactify
$\mathcal{U} \subseteq \overline{\mathcal{U}}^{KSBA}$, the compactification is a proper DM stack

② $-(K_X+D)$ ample $\rightsquigarrow$ K stability gives $\mathcal{U} \subseteq \overline{\mathcal{U}}^{K}$, this is an algebraic stack with projective good moduli space

③ What happens if $K_X + D \sim_{\mathbb{Q}} 0$?

Used techniques: GIT, KSBA + ε; today: new approach

The objects : (X, D)

① X projective ② $K_X + D \sim_{\mathbb{Q}} 0$ ③ (X, D) slc

④ $-K_X$ ample

EG: • Y degree 2 K3, $Y \longrightarrow \mathbb{P}^2$ ramified along a sextic

• $\left\{ \left(\mathbb{P}^n, \frac{n+1}{d} H \right) \right\}$

• (X, L) polarized cy $\rightsquigarrow \left(C(X, L), C_\infty(X, L) \right)$

Today: $\mathcal{U} = \left\{ \left(\mathbb{P}^2, \frac{3}{d} C \right) : C \text{ smooth of degree } d \right\}$

Other examples of compactifications of $\{ (\mathcal{U}, C) \}$

(i) Ascher, De Vleming, Liu $\quad \mathcal{U} = \{(\mathbb{P}^2, cC)\}$

$c < \frac{3}{d}$

Using K-stability : $\mathcal{U} \subseteq \overline{\mathcal{U}}_c^K$ $\quad \overline{\mathcal{U}}_c^K$ admits a projective good moduli space

(ii) Hacking : $\mathcal{U} = \{ (\mathbb{P}^2, cC) \}$ $\quad \frac{3}{d} < c < \frac{3}{d} + \varepsilon$

Using KSBA : $\mathcal{U} \subseteq \overline{\mathcal{U}}^{KSBA}$, proper DM stack

For our problem :

$$\mathcal{D}_d^{CY}(\mathbb{C}) = \left\{ \begin{array}{l} \text{slc Fano } (X, D) : K_X + D \sim_{\mathbb{Q}} 0 \\ \text{that admit } \mathbb{Q}\text{-Gorenstein smoothing} \\ (\mathbb{P}^2, \tfrac{3}{d} C) \end{array} \right\}$$

$$\mathcal{P}_d^{CY}(\mathbb{C}) = \left\{ \begin{array}{l} \text{slc Fano } (X, D) : K_X + D \sim_{\mathbb{Q}} 0 \\ \text{that admit } \mathbb{Q}\text{-Gorenstein smoothing} \\ \qquad (\mathbb{P}^2, \tfrac{3}{d} C) \end{array} \right\}$$

① Objects have ∞ automorphisms

③ $\mathcal{P}_d^{CY}$ is <u>not</u> bounded: $(\mathbb{P}(a^2, b^2, c^2), \pi)$

$$a^2 + b^2 + c^2 = 3abc$$

<u>Thm</u>:

i) $\exists$ an algebraic stack $\mathcal{U} \subseteq \mathcal{P}_{d,m}^{CY} \subseteq \mathcal{P}_d^{CY}$:

$\mathcal{P}_{d,m}^{CY}$ bounded, admits a projective good moduli space its points param S-equivalence classes of pairs in $\mathcal{P}_d^{CY}$

Thm:

i) $\exists$ an algebraic stack $\mathcal{U} \subseteq \mathcal{P}^{CY}_{d,m} \subseteq \mathcal{P}^{CY}_{d}$:

$\mathcal{P}^{CY}_{d,m}$ bounded, admits a projective good moduli space
the pts of $P^{CY}_{d,m}$ param S-equivalence classes of pairs in $\mathcal{P}^{CY}_{d}$

ii) $$\overline{\mathcal{U}}^{K}_{\frac{3}{d}-\varepsilon} \longrightarrow \mathcal{P}^{CY}_{d,m} \longleftarrow \overline{\mathcal{U}}^{KSBA}_{\frac{3}{d}+\varepsilon}$$

iii) $\mathcal{P}^{CY}_{d,m} \longrightarrow P^{CY}_{d,m}$, λ_{Hodge} is ample on $P^{CY}_{d,m}$

Rmk: Recall $\mathcal{M}$ has a good moduli space if

$$\begin{tikzcd} \mathcal{M} \arrow[r] & M \\ {[\mathrm{Spec}(A)/GL]} \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & \mathrm{Spec}(A^{GL}) \arrow[u, "\text{ét}"'] \end{tikzcd}$$

alg space

$\mathcal{M} = [V^{ss}/GL_n]$

$M = V/\!/GL_n$

Two pairs (X_1, D_2) (X_2, D_2) are s-equivalent if $\exists$

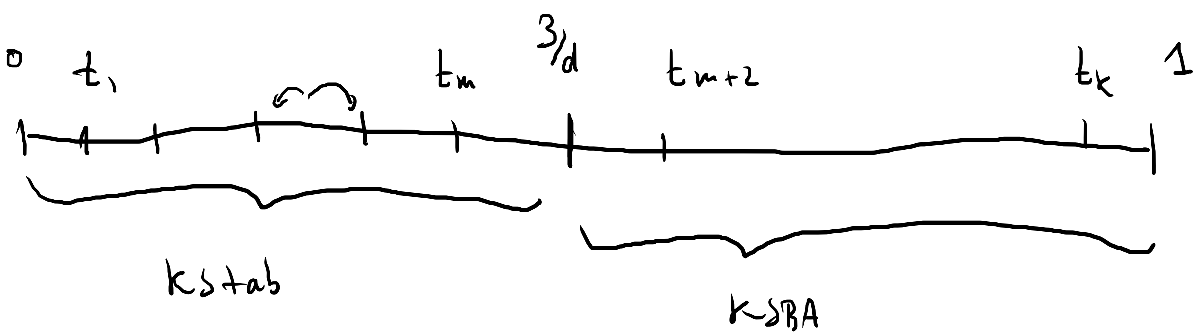
$$(\mathcal{X}_i, \mathcal{D}_i) \xrightarrow{\pi} \mathbb{A}^1$$

① $\mathbb{G}_m \curvearrowright (\mathcal{X}_i, \mathcal{D}_i)$: π is equiv

② $(\mathcal{X}_i, \mathcal{D}_i)_1 = (X_i, D_i)$

③ $(\mathcal{X}_1, \mathcal{D}_1)_0 \cong (\mathcal{X}_2, \mathcal{D}_2)_0$

② We have wall-crossings in K & KSBA stability:



$$\begin{tikzcd} (\mathcal{X}, \mathcal{B}) \arrow[d, "\pi"] \\ \mathcal{P}^{CY}_{d,m} \arrow[d, "f"'] \\ P^{CY}_{d,m} \end{tikzcd}$$

1. $L_{\mathcal{X}} = \pi_* \left(\mathcal{O} (K_\pi + \mathcal{B})^{\otimes m} \right)$ is a line bundle
2. $\exists$ $\lambda_{\mathcal{H}}$ line bundle on $P^{CY}_{d,m}$:
$$L_{\mathcal{H}}^{\otimes d} = f^* \lambda_{\mathcal{H}}^{\otimes d}$$
3. $\lambda_{\mathcal{H}}$ ample

Sketch:

Recall: $\mathcal{M}$ alg stack

$\left.\begin{array}{l} \text{1. } \Theta\text{-reductive} \\ \text{2. } S\text{-complete} \\ \text{3. of finite type} \end{array}\right\} \Rightarrow$ has a good moduli space

② $\mathcal{P}_d^{CY}$ not bounded $\iff$ $3 \mid d$ } done if $3 \nmid d$.

$\mathcal{P}_d^{CY}$ is Ⓗ red & S-complete

③ If $3 \mid d$:

$$\mathcal{P}_{d,m}^{CY} = \{ (X, D) \in \mathcal{P}_d^{CY} : \mathrm{id}_X(K_X) \leq m \}$$

↳ These are Ⓗ-red & S-complete

↳ for $m \gg$ $\mathcal{P}_{d,m}^{CY}$ are proper

- all the pairs that are strictly l.c. admit 1-complement
- Special adjunction

② Automatic $\mathcal{P}_{d,m}^{CY}$

③ Requires understanding all S-equivalence classes in $\mathcal{P}_d^{cy}$.

In general:

$$\mathcal{M}_{\chi, N, \vec{c}} = \{ (X, D) : N(K_X + c) \sim 0, \ \chi(K) = \chi, \text{coeff of } D \text{ in } \vec{c} \}$$

↳ Ⓗ-red & S-complete